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ASYMPTOTIC ANALYSIS OF NONAXISYMMETRIC EQUILIBRIUM MODES OF A THIN SHALLOW SPHERICAL SHELL*

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The stability of a thin, elastic, spherical shell with absolutely fixed support contour loaded by a uniform external pressure is examined within the framework of perturbation theory in the Koiter—Fitch form /1,2/. The state of stress and strain, the stability, and the bifurcation of the equilibrium modes for which the lifting capacity of the shell is not exhausted are investigated.

The analysis is limited to pressures of the form

$$p(r) = \delta\eta(r) + p$$

Here δ is a small numerical parameter, $\eta(r)$ is a function of the polar radius r that characterizes the pressure distribution over the shell meridian, and p is a scalar parameter on the order of one, where among its numerical values a sequence of eigenvalues of the appropriate nonlinear boundary value problem linearized in the neighborhood of the axisymmetric solution is considered.

As is known, substantial discrepancies are observed between the upper critical pressures obtained according to a geometrically nonlinear theory and the data of precision experiments. The critical pressures determined experimentally are, as a rule, below the first eigenvalue (in absolute value). The lower critical pressures are obtained because of geometric imperfections in the middle surface, the formation of domains in the shell in which physically nonlinear phenomena are essential, the influence of a "wall-thickness-variation" factor, etc.

Data (see /3—7/, say) are also known that show that the critical pressures can exceed the corresponding results of theoretical investigations. Such results are obtained if the loading is quasistatic, the deviations in the radius of curvature do not exceed 0.01% at separate points of the shell surface, and variations in the shell thickness do not exceed 1.5%.

It should be noted that there are significant discrepancies between the data of precision experiments of different authors. Firstly, a spread in the critical pressures is observed, that reaches 20% in a number of cases. Secondly, some researchers observed nonaxisymmetric buckling modes under loading /3,4,7/, and others only axisymmetric modes /6/. These and analogous experiments permitted the advancement of several hypotheses.

1°. Since the critical pressures corresponding to shell snap are determined during the experiments /3,4/, then a disagreement between the first eigenvalue p^* and the pressure p° at which snap occurs, probably holds, i.e., nonaxisymmetric equilibrium modes branch off in the neighborhood of the bifurcation point p^*, but the lifting capacity of the shell is not exhausted /8/. In this case the shell can perceive a pressure exceeding the first eigenvalue. However, it is shown in /9/ that $p^\circ = p^*$ holds in the problem under study for those values of the geometric parameters at which nonaxisymmetric bifurcation is possible.

2°. According to /10,11/, only axisymmetric solutions branch off in the neighborhood of the bifurcation points (**). In this case, the critical pressures in the precision experiments should agree with the corresponding branch points of the boundary value problem of nonlinear shell theory in an axisymmetric formulation, and their existing defects are reduced somewhat under real conditions. As a rule the results of /11/ are used for a foundation of this viewpoint.

It is shown below that the stability of a thin shell is responsive to the form of the function $\eta(r)$. For a given fixing of the support contour, the shell buckles in a snapping mode for some kinds of functions $\eta = \eta_1(r)$, bifurcation is observed for other kinds $\eta = \eta_2(r)$, but the shell carrying capacity is not exhausted. A function $\eta = \eta_3(r)$ is determined for which branching off of the nonaxisymmetric modes from the axisymmetric solution occurs without snapping for all the nonaxisymmetric bifurcation points. Under real conditions the distribution of the perturbing pressure depends on the structural features of the experimental apparatus, the loading method, etc. One of the reasons for the discrepancy in experimental data can

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**) Pogorelov, A.V., On spherical shell buckling modes. Dokl. Vses. Konf. po Teorii Obolochek i Plastin (Report to All-Union Conf. on Theory of Plates and Shells), Rostov-on-Don, 1971.

therefore be the fact that different critical pressures, corresponding to different buckling modes, are realized in experimental investigations.

1. Formulation of the problem. The equilibrium and strain compatibility equations of the geometrically nonlinear theory of a shallow, elastic, spherical shell whose middle surface is identified with a plane, have the following form in dimensionless variables

$$\mu\Delta^2 w = L(w, \Phi) + \theta\Delta\Phi + \rho(r), \quad \mu\Delta^2\Phi = -{}^1/_2 L(w, w) - \theta\Delta w, \quad \mu = h / a\gamma \ll 1 \tag{1.1}$$

$$L(w, \Phi) = w''(\Phi'/r + \Phi^{\cdot\cdot}/r^2) + \Phi''(w'/r + w^{\cdot\cdot}/r^2) - 2(\Phi^{\cdot}/r - \Phi^{\cdot\prime})(w^{\cdot}/r - w^{\cdot\prime})/r^2$$

$$(\ )^{\cdot} = \partial/\partial\varphi, \quad (\ )' = \partial/\partial r, \quad \gamma^2 = 12(1 - \nu^2)$$

Here w is the normal dispalcement, Φ is the Airy stress function, $\rho(r)$ is the external pressure, h is the shell thickness, a is the planform radius, ν is the Poisson's ratio, $\theta = a/R$ is half the shell aperture, R is the radius of curvature, and (φ, r) are the polar coordinates.

The dimensionless quantities in (1.1) are related to the dimensional quantities marked with the subscript d by means of the formulas (E is Young's modulus):

$$\rho_d(r_d) = Eh^2\rho(r)/(a^2\gamma), \quad \Phi_d = Eha\Phi/\gamma, \quad w_d = aw, \quad r_d = ar$$

We supplement the system (1.1) by the boundary conditions

$$\text{a) } r = 1, \quad w = w' = 0, \quad \Phi'' - \nu(\Phi' + \Phi^{\cdot\cdot}) = 0 \tag{1.2}$$

$$\Phi''' - \nu(\Phi'' - \Phi' + \Phi^{\cdot\cdot\prime} - 2\Phi^{\cdot\cdot}) + 2(1 + \nu)(\Phi^{\cdot\cdot\prime} + \Phi^{\cdot\cdot}) + \nu\Phi - \Phi' - \Phi^{\cdot\cdot} = 0$$

$$\text{b) } r = 1, \quad w = w' = 0, \quad \Phi' + \Phi^{\cdot\cdot} = \Phi^{\cdot\prime} - \Phi^{\cdot} = 0$$

Conditions a) and b) correspond to rigid clamping of the support contour and sliding clamping.

The energy functional of an elastic shallow spherical shell has the form

$$\Pi = \frac{1}{2}\int_0^{2\pi} d\varphi \int_0^1 [\sigma_r\varepsilon_r + \sigma_\varphi\varepsilon_\varphi + 2\sigma_{r\varphi}\varepsilon_{r\varphi} + M_r\chi_r + M_\varphi\chi_\varphi + 2M_{r\varphi}\chi_{r\varphi}]\, r\,dr \tag{1.3}$$

$$\varepsilon_r = u_1' + \theta r w' + {}^1/_2(w')^2, \quad \chi_r = -w''$$

$$r\varepsilon_\varphi = u_1 + u_2^{\cdot} + {}^1/_2(w^{\cdot})^2/r, \quad r\chi_\varphi = -w' - w^{\cdot\cdot}/r$$

$$2r\varepsilon_{r\varphi} = u_1^{\cdot} + ru_2' - u_2 + \theta r w^{\cdot} + w^{\cdot}w', \quad \chi_{r\varphi} = -(w^{\cdot}/r)'$$

$$r\sigma_r = \Phi' + \Phi^{\cdot\cdot}/r, \quad \sigma_\varphi = \Phi'', \quad r\sigma_{r\varphi} = \Phi^{\cdot}/r - \Phi^{\cdot\prime}$$

$$\varepsilon_r = \mu(\sigma_r - \nu\sigma_\varphi), \quad \varepsilon_\varphi = \mu(\sigma_\varphi - \nu\sigma_r), \quad \varepsilon_{r\varphi} = \mu(1 + \nu)\sigma_{r\varphi}$$

Here $\{u_1, u_2\}$ are tangential displacements, and $M_r, M_\varphi, M_{r\varphi}$ are radial, circumferential, and twisting moments.

2. Method of solution. Let us assume that in the continuation of the solution in the pressure density parameter δ, the vector of the solution $V = \{w, \Phi\}$ and δ can be expanded in series in integer powers of ξ in the neighborhood of the branchpoint

$$V(r, \varphi, \delta) = V_0(r, \delta) + \sum_{n=1}^{\infty} \xi^n V_n(r, \varphi), \quad \delta = \sum_{n=1}^{\infty} \xi^n \delta_n \tag{2.1}$$

where the representation (2.1) is valid at least in the asymptotic sense as $\xi \to 0$. Here $V_0(r, \delta)$ is the axisymmetric mode from which the solutions branch off and contain small non-axisymmetric terms $\xi^n V_n$. For each of the boundary conditions (1.2), the vector-function $\{w_0, \Phi_0\}$ is defined by the relationship

$$V_0(r, \delta) \equiv \left\{\int_1^r \beta(t, \delta)\, dt, \ \int_0^r \psi(t, \delta)\, dt\right\}$$

Here $\{\beta, \psi\}$ satisfies the nonlinear boundary value problems for the ordinary differential equations

$$\mu A_r\beta = r\theta\psi + \beta\psi + \int_0^r \rho(t)\, t\, dt, \quad \mu A_r\psi = -r\theta\beta - {}^1/_2\beta^2 \tag{2.2}$$

$$\beta(0) = 0, \quad \psi(0) = 0, \quad A_r(\ ) = r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}r(\ )\right]$$

$$\text{a) } r = 1, \quad \beta = \psi' - \nu\psi = 0; \quad \text{b) } r = 1, \quad \beta = \psi = 0$$

Substituting the expansion (2.1) into (1.1) and the boundary conditions (1.2), and then sequentially retaining terms of the first and second degree in ξ, we obtain boundary value problems to determine $\mathbf{V}_1$ and $\mathbf{V}_2$. Expanding $\mathbf{V}_1$ into a Fourier series and making the change of variable $\mathbf{V}_{1,n}(r) = r^n\mathbf{Z}_n(r)$ in each coefficient of the Fourier series for $\mathbf{V}_{1,n}$, we obtain that $\mathbf{Z}_n(r) = \{W_n, F_n\}$ will be eigenvector-functions of the spectral problems

$$\mu\Delta_n^2 W_n = T_r^n F_n + \Gamma_r^n W_n, \quad \mu\Delta_n^2 F_n = -T_r^n W_n \tag{2.3}$$
$$\mathbf{Z}_n^k = 0, \quad r = 0, \quad k = 1, 3$$
$$\text{a) } r = 1, \quad W_n = W_n' = (c_r^n + \omega_{\nu r}^n)F_n = M_\nu^n F_n = 0$$
$$\text{b) } r = 1, \quad \mathbf{Z}_n = \mathbf{Z}_n' = 0$$
$$\Delta_n = (\ )'' + (2n+1)/r, \quad T_r^n = \theta\Delta_n + \beta' E_r^n + \beta c_r^n$$
$$\Gamma_r^n = \psi c_r^n + \psi' E_r^n, \quad c_r^n = [(\ )'' + 2n/r + (n^2 - n)/r]/r$$
$$E_r^n = (n - n^2)/r^2 + (\ )'/r, \quad \omega_{\nu r}^n = \nu\,[n(n-1) - (\ )']$$
$$M_\nu^n = (\ )''' + 3n(\ )'' + (n^2 - 1)(1 - \nu) - 3n + n(1 - n^2)$$
$$(1 + \nu)$$
$$\| W_n \|_{c^\circ} = \mu, \quad n = 2, 3, \ldots, N$$

The pressure on the shell outer surface p, which enters implicitly into (2.3) in terms of the components of the vector function $\{\beta, \psi\}$ is the spectral parameter in eigenvalue problems.

Let us substitute (2.1) into (1.1) and (1.2) and require that the residual be on the order of $O(\xi^\alpha)$, $\alpha > 2$, as $\xi \to 0$ in satisfying the equations and boundary conditions. We hence obtain inhomogeneous partial differential boundary value problems for V_2. Analysis of these latter shows that if $p \in \{p_n\}$, where p_n is an eigenvalue of the problem (2.3), then $\mathbf{V}_2$ for each n is representable in the form

$$\mathbf{V}_{2n}(r, \varphi) = r^{2n}\mathbf{G}_n(r)\cos 2n\varphi + \int_0^r \mathbf{H}_n(t)\,dt, \quad \mathbf{G}_n(r) \equiv \{\tau(n, r), \omega(n, r)\}, \quad \mathbf{H}_n(r) = \{g(n, r), f(n, r)\}$$

where $\{\tau, \omega\}$, $\{g, f\}$ is the solution of the following boundary value problems

$$\mu\Delta_{2n}^2\tau = T_r^{2n}\omega + \Gamma_r^{2n}\tau + \alpha, \quad \mu\Delta_{2n}^2\omega = -T_r^{2n}\tau + \zeta, \quad \mathbf{G}_n^{(k)}(0) = 0, \quad k = 1, 3 \tag{2.4}$$
$$\text{a) } r = 1, \quad \tau(n, r) = \tau'(n, r) = M_\nu^{2n}\,\omega(n, r) = (c_\nu^{2n} + \omega_{\nu r}^{2n})\omega(n, r) = 0, \quad \text{b) } r = 1, \quad \mathbf{G}_n = \mathbf{G}_n' = 0$$

$$\mu A_r g = \theta r f + \beta f + \psi g + \Omega, \quad \mu A_r f = -(\theta r + \beta)g + S, \quad g(0) = f(0) = 0 \tag{2.5}$$
$$\text{a) } r = 1, \quad g = f' - \nu f = 0, \quad \text{b) } r = 1, \quad \mathbf{H}_n = 0$$
$$\alpha = {}^1/_2\,\{lr^{-2}[F_n W_n'' - F_n W_n' r^{-1} - W_n(F_n' r^{-1} - F_n'')] +$$
$$4nr^{-2}W_n'F_n' + r^{-1}(W_n''F_n' + F_n''W_n')\} + n^2r^{-2}F_n'W_n'$$
$$\zeta = -r^{-2}[(W_n')^2 n(1 + n/2) + lW_n(W_n'' - r^{-1}W_n')/2 + rW_n'W_n''/2]$$
$$\Omega = r^{2n}\{W_n'F_n' + l[F_nW_n' + W_nF_n']/r + 2nr^{-2}lF_nW_n\}/2$$
$$S = -r^{2n}\{2nlr^{-2}W_n^2 + (W_n')^2 + 2lr^{-1}W_n'W_n\}/4, \quad n = 2, 3 \ldots N, \quad l = n - n^2$$

Therefore, the problem (1.1), (1.2) formulated is reduced to a recurrent sequence for each of the boundary conditions a) and b), consisting of the boundary value problems 1) for the nonlinear ordinary differential equations (2.2), 2) the eigenvalue problems (2.3), and 3) for the systems of linear inhomogeneous equations (2.4) and (2.5) in which the solution of the problem (2.2) enters into the coefficients and the eigenvector-function $\{W_n, F_n\}$ into the inhomogeneous parts α, ζ, Ω, S.

Remarks. 1°. The solutions $\mathbf{G}_n(r)$, $\mathbf{H}_n(r)$ are known to the accuracy of a constant that is determined by the Liapunov—Schmidt method.

2°. Let E_2 be the Hilbert space of the two-dimensional vector functions $x = (x_1, x_2)$, $y = (y_1, y_2), \ldots$ with the scalar product

$$\langle x, y\rangle_{E_2} = \int_0^{2\pi} d\varphi \int_0^1 (x_1y_1 + x_2y_2)\,r\,dr$$

The scalar product $\langle \mathbf{V}_1, \mathbf{V}_j\rangle$ equals zero for $j \geqslant 2$. Therefore, the expression

$$\xi = \langle \mathbf{V} - \mathbf{V}_0, \mathbf{V}_1\rangle / \langle \mathbf{V}_1, \mathbf{V}_1\rangle$$

can be a formal definition of the small parameter ξ.

Let us investigate the power series (2.1) for δ under certain constraints. Let the shell be subjected to the pressure

$$\rho(r) = p + \delta\eta(r)$$

where p is the uniform external pressure equal to one of the eigenvalues (2.3), and $\eta(r)$ is a sufficiently smooth function satisfying the condition

$$0 < \left| \int_0^1 g(r) \left[\int_0^r t\eta(t)\,dt \right] dr \right| < \infty$$

Let $\sigma, \varepsilon, \mathbf{U}$, be, respectively, the generalized stress, strain, and displacement, and let L_1 and L_2 be linear and quadratic differential operators in $\mathbf{U}$. Then within the framework of the geometrically nonlinear theory in a "quadratic" approximation, the function ε and its variation $\delta\varepsilon$ take the form

$$\varepsilon = L_1(\mathbf{U}) + {}^1/_2 L_2(\mathbf{U}), \quad \delta\varepsilon = L_1(\delta\mathbf{U}) + L_{11}(\mathbf{U}, \delta\mathbf{U}) \tag{2.6}$$

Here L_{11} is a bilinear differential operator. Let us expand $\mathbf{U}$ in a series analogous to (2.1). From (2.6) we have

$$\varepsilon = \varepsilon_0 + \xi[L_1(\mathbf{U}_1) + L_{11}(\mathbf{U}_0, \mathbf{U}_1)] + \xi^2[L_1(\mathbf{U}_2) + L_{11}(\mathbf{U}_0, \mathbf{U}_2) + {}^1/_2 L_2(\mathbf{U}_1)] + \xi^3[L_1(\mathbf{U}_3) + L_{11}(\mathbf{U}_1, \mathbf{U}_2) + \tag{2.7}$$

$$L_{11}(\mathbf{U}_0, \mathbf{U}_3) + \ldots] + O(\xi^4), \quad \delta\varepsilon = \delta\varepsilon_0 + \sum_{k=1}^{n} \xi^k L_{11}(\mathbf{U}_k, \delta\mathbf{U}) + O(\xi^4)$$

We convert (2.7) by replacing $\mathbf{U}_0(p+\delta\eta)$ by a segment of its Taylor series in the neighborhood of the point $p \in \{p_n\}$. We then obtain

$$\varepsilon = \varepsilon_0 + \xi\varepsilon_1 + \xi^2\varepsilon_2 + \xi^3\varepsilon_3 + O(\xi^4) \tag{2.8}$$

$$\varepsilon_0 = L_1(\mathbf{U}_0) + {}^1/_2 L_2(\mathbf{U}_0), \quad \varepsilon_1 = L_1(\mathbf{U}_1) + L_{11}(\mathbf{U}_0, \mathbf{U}_1)$$

$$\varepsilon_2 = L_1(\mathbf{U}_2) + \delta_1 L_{11}(\eta\mathbf{U}_{0,\rho}^{(1)}, \mathbf{U}_1) + L_{11}(\mathbf{U}_0, \mathbf{U}_2) + {}^1/_2 L_2(\mathbf{U}_1)$$

$$\varepsilon_3 = L_1(\mathbf{U}_3) + \delta_2 L_{11}(\eta\mathbf{U}_{0,\rho}^{(1)}, \mathbf{U}_1) + {}^1/_2 \delta_1^2 L_{11}(\eta^2\mathbf{U}_{0,\rho}^{2}, \mathbf{U}_1) +$$
$$\delta_1 L_{11}(\eta\mathbf{U}_{0,\rho}^{(1)}, \mathbf{U}_2) + L_{11}(\mathbf{U}_1, \mathbf{U}_2) + L_{11}(\mathbf{U}_0, \mathbf{U}_3)$$

Here $(\ )_\rho^{(k)}$, $k = 1, 2$ are Fréchet derivatives of order k with respect to ρ.

Let us assume that the equilibrium of the generalized stress σ and the pressure p is assured by the condition

$$\int_S \sigma\delta\varepsilon\,ds = \int_S p\delta\mathbf{U}\,ds \tag{2.9}$$

Here S is the shell middle surface. Then by using the Hooke's law $\sigma_i = \Gamma\varepsilon_i$, $i = 0, \ldots 3$, the kinematic and static relationships from (1.3) and (2.7)–(2.9), we group the terms for ξ^n, $n = 0, \ldots 3$ sequentially in (2.9). We obtain a variational formulation of the problems (2.2) and (2.3) in a zero-th and first approximation. We expand σ_0 and $\mathbf{U}_0$ in a Taylor series in powers of $\eta(r)\delta$. We replace $\delta\mathbf{U}$ by $\mathbf{U}_1$, and taking into account that

$$\int_S \sigma_1\varepsilon_j\,ds = \int_S \sigma_j\varepsilon_1\,ds, \quad L_{11}(\mathbf{U}_j, \mathbf{U}_j) = L_2(\mathbf{U}_j), \ j = 1, 2, 3$$

we find condition (2.9) in the asymptotic form

$$\sum_{n=2}^{\infty} A_n\xi^n = 0 \tag{2.10}$$

$$A_2 = \int_S \left\{\delta_1 B_1 + \sigma_1\left[\frac{3}{2}L_2(\mathbf{U}_1) + L_1(\mathbf{U}_2) + L_{11}(\mathbf{U}_0, \mathbf{U}_2)\right] + \sigma_0 L_{11}(\mathbf{U}_2, \mathbf{U}_1)\right\} ds$$

$$A_3 = \int_S \{\delta_2 B_1 + \sigma_1[L_{11}(\mathbf{U}_1, \mathbf{U}_2) + L_1(\mathbf{U}_3) + L_{11}(\mathbf{U}_0, \mathbf{U}_3)] + \sigma_0 L_{11}(\mathbf{U}_3, \mathbf{U}_1) + \sigma_2 L_2(\mathbf{U}_1)\}\,ds$$

$$B_1 = \sigma_{0,\rho}' \eta L_2(\mathbf{U}_1) + 2\sigma_1 L_{11}(\mathbf{U}_1, \eta\mathbf{U}_{0,\rho}')$$

Equating the coefficients of ξ^2 and ξ^3 to zero in (2.10) and using the variational formulation of the eigenvalue problem, we determine the first two terms of the expansion (2.1) for δ in the form

$$\delta_1 = 0, \quad \delta_2 = J / J_0, \quad J = \int_0^1 \left[fS - g\Omega + \frac{1}{2} r^{4n+1} (\alpha\tau - \zeta\omega) \right] dr, \quad J_0 = \int_0^1 \left[g(r) \int_0^r t\eta(t)\, dt \right] dr \qquad (2.11)$$

The expression for J_0 has been found here by using identity transformations of the problem (2.5) and the problem obtained by differentiating the equations and boundary conditions in (2.2) with respect to the parameter δ.

It follows from (2.11) that three cases are possible for a δ-neighborhood of the bifurcation point p^*.

a) $\delta_2 > 0$. In this case the nonaxisymmetric solution exists for $\delta > 0$, i.e., the shell is able to sustain the pressure $p > p^*$, and the nonaxisymmetric equilibrium mode can, if it is energetically suitable, be observed in the static state.

b) $\delta_2 < 0$. Bifurcation of the nonaxisymmetric mode is accompanied by snapping and it is impossible to observe the mode branching off in a static formulation.

c) $\delta_2 = 0$. To analyze buckling, the last terms in the series (2.1) for δ must be taken into account.

It follows from (2.11) that only the solution of the problems (2.3)—(2.5) enters explicitly into J, while J_0 depends also on $\eta(r)$, i.e., on the loading method, the structural features of the apparatus being used, etc. If $g(r)$ is a sign-variable function, then the sign of δ_2, meaning also the buckling mode, depends on the distribution of a small perturbing pressure $\delta\eta(r)$ over the shell opening in the neighborhood of p^*.

3°. For numerical integration, the boundary value problem (2.2) was reduced to two Cauchy problems $(v \equiv \{\beta(r), \psi(r)\})$

$$1)\ r \in (0, {}^1/_2),\ v_+'' = f(v_+', v_+, r, \theta, \mu, \rho),\ r = 0,\ v_+ = 0,\ v_+' = \{s_1, s_2\}$$
$$2)\ r \in ({}^1/_2, 1),\ v_-'' = f(v_-', v_-, r, \theta, \mu, \rho)$$
$$a)\ r = 1,\ v_- = \{0, s_3\},\ v_-' = \{s_4, \nu s_3\};\ b)\ r = 1,\ v_- = 0,\ v_-' = \{s_4, s_3\}$$

Here f is a two-dimensional vector function representing the right sides of the system of differential equations solved for the highest derivatives, $s_1 \ldots s_4$ are unknown alignment parameters, v_+ and v_- denote the function v for $0 \leqslant r < {}^1/_2$ and ${}^1/_2 < r \leqslant 1$, respectively. The problems 1) and 2) were integrated by the Runge—Kutta method. The alignment parameters are found by the Newton method from the adjoint conditions

$$r = 0.5,\ \varepsilon \ll 1,\ \| v_+^{(k)}(s_1, s_2) - v_-^{(k)}(s_3, s_4) \|_{R^2} < \varepsilon, \quad k = 0, 1$$

The linear problems (2.3)—(2.5) were solved analogously. The S. G. Godunov procedure for raising the accuracy of the computation /12/ was hence used to integrate the Cauchy problems of the type 1) and 2) for the systems (2.3), (2.4). The quantity of points at which the Gram—Schmidt orthogonalization was performed was eight. Their coordinates $r_i\ (i = 1 \ldots 8)$ were determined automatically from the condition $\| W_n(r_i) \|_{C^0} > m\ (m = 0.3)$. In analyzing the problems (2.3)—(2.5) a system of linearly-independent vectors was constructed and then the general solution was found from a linear combination. The eigenvalue was determined from the vanishing of the appropriate determinant.

4°. As $r \to 0$, the coefficients of the equations have a singularity, hence for $r \in [0, \Delta r]$ the solution of the problem (2.2)—(2.5) was replaced by a segment of a Taylor series. The quantity Δ_r varied between 0.11—0.27.

3. Numerical results.

It was assumed $\nu = 0.3$. in the computations. The functions $g^*(r) = 1.66 \cdot 10^3\, g(r)$ and $g^{**}(r) = 3.33 \cdot 10^2\, g(r)$ for rigid and moving clamping are presented respectively, in Figs.1 and 2. The numbers of the curves correspond to the number of the eigenvalues. Curve *4* in Fig.1 corresponds to values of the parameters $\lambda = 8.52$, $p = 0.76$, curve *8* to $\lambda = 14$, $p = 0.78$ (here and henceforth, $\lambda^2 = \theta / \mu$). In Fig.2 $n = 4$, $\lambda = 11$, $p = 0.34$; $n = 6$, $\lambda = 16$, $p = 0.32$; $n = 8$, $\lambda = 19$, $p = 0.30$. It is seen that for the boundary conditions being considered $g(r)$ is an oscillating function for which the number of zeroes will be the greater, the greater the λ. For large λ (see curves *6* and *8* in Fig.2, for example), the function $g(r)$ has a definite edge effect, in particular, the points of maximal and minimal values shift to the support contour as λ grows. Therefore, in the neighborhood of the eigenvalues of the problem (2.3), the spherical shell is quite responsive to the kind of perturbing pressure. Two domains exist in a shell (this is the neighborhood of the points A and B for the meridian being considered) at which the change in $\eta(r)$ affects the buckling method especially strongly. For thin shells the behavior of $\eta(r)$ is essential primarily in the edge effect zone.

The dependence of the Koiter parameter $b = 5 \cdot 10^5\, \delta_2$ on λ is shown in Fig.3 for the problem (2.3), b) - (2.5),b). The numbers of the curves here correspond to the number of the eigenvalue $n, \eta(r) \equiv 1$. It is seen that b is a negative continuous function of λ for fixed n. The function $b(\lambda)$ is multivalued, and its branches are determined by selecting n. The transition from one branch to another in the domain under investigation is possible only by a jump whose minimal value diminishes as λ grows. It follows from the results presented in Fig.3 that shells with moving clamped support contour buckle by snapping when $7.6 \leqslant \lambda \leqslant 21$.

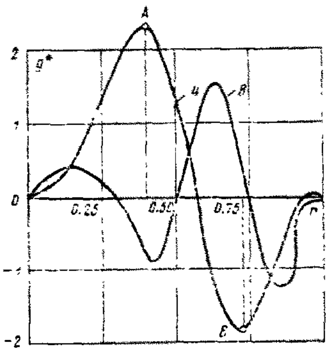

Fig.1

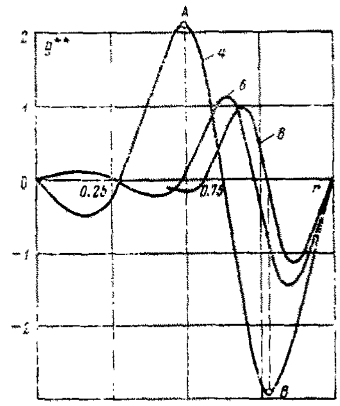

Fig.2

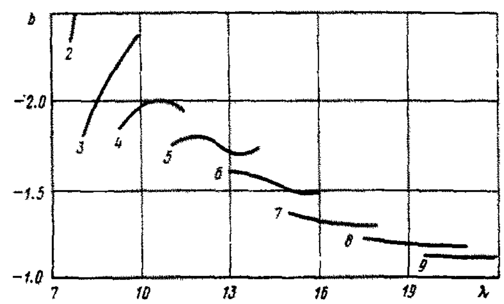

Fig.3

The deflections $\pi_n = 10^5 e_n r^n W_n$ ($e_8 = {}^5/_2$, $e_6 = {}^5/_{12}$, $e_4 = {}^5/_4$) are presented in Fig.4. The clamping conditions and numerical value of n, λ, p are the same here as in the example illustrated in Fig.2.

An assumption that the relationship $|w_\varphi''| \ll |w_r'|$ holds in the support contour zone of width $O(\mu\,|\ln \mu|)$ is used in a number of cases in an asymptotic analysis of the Marguerre—Vlasov equations. Its verification at some point of the edge effect zone, for instance at the point C, shows that $|w_{\varphi\varphi}''| > |w_r'|$. Therefore, the assumption noted can induce a significant error into the asymptotic analysis of shells with a finite, albeit large λ ($\lambda \sim 20$).

The eigenvalue distribution of the problem (2.3),a) is characterized by the dependence $n(p_n)$ presented in Fig.5. The solid lines are the data of a numerical computation, and the dashed line is the result of asymptotic integration of the spectral problem (see Sects.1 and 2). No constraint is imposed here in the asymptotic analysis on the quantity of waves in the circumferential direction /13/(*). The curves marked with the numbers $1, 2, 3$ correspond to the values $\lambda = 12, 15, 18$. It follows from the results presented in Fig.5 that rapidly and weakly oscillating waves, described by two branches of the function $n(p_n)$ can appear on the shell. Here the quantity of waves corresponding to one branch grows with the rise in pressure and that of the other branch decreases. Small nonaxisymmetric equilibrium modes are constructed for each p_n by the Liapunov—Schmidt method. It turns out that if p_n is a simple eigenvalue, then two nonaxisymmetric equilibrium modes containing an identical number of modes in the circumferential direction but with the points of their maximal normal displacements shifted in the circumferential direction by the phase $\alpha = \pi / n$, branch off from the axisymmetric mode. Investigations of the nonaxisymmetric modes for each point of the spectrum $\{p_n\}$ showed

*) Larchenko, V. V., Nonlinear stability and estimate of the efficiency of the asymptotic method in elastic spherical shells for different boundary conditions. Summary of Kandidat Dissertation, Rostov-on-Don, 1977.

that for $\lambda \leqslant 22$ the quantity of nonaxisymmetric solutions branching off which have a different number of modes will grow nonmonotonically as λ increases, and is determined for large λ by the asymptotic formula

$$\lambda \to \infty, \quad N \sim [1.33\lambda]$$

where $[\cdot]$ is the integer part of the number. If λ is fixed, and p is a multiple value and belongs to the beginning of the spectrum $\{p_n\}$, then not more than two modes containing a different number of nonaxisymmetric modes branch off from the axisymmetric solution. Here if

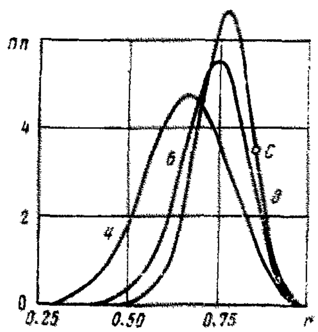

Fig.4

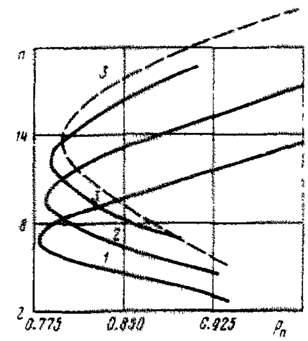

Fig.5

$p = \min_n \{p_n\}$, then the numbers of waves in the circumferential direction that are referred to the two different modes, differ by one. If the first eigenvalue is simple, and λ is sufficiently large, then the corresponding quantity of waves can be obtained by an asymptotic method. This confirms the comparison between numerical /13,14/ and asymptotic results. This latter results in an asymptotic formula for the quantity of waves of the form

$$\lambda \to \infty, \quad n \sim [0.808\lambda]$$

Comparing the asymptotic results obtained with the results of the nonlinear theory in a nonaxisymmetric formulation shows that for all λ for which the nonaxisymmetric bifurcation holds, the first critical pressure of the problem (2.3),a) is determined by an asymptotic method with less than 6% error.

Let us note that the difference between p_n and p_{n+1} at the beginning of the spectrum is, as a rule, considerably less than the analogous quantity far from the first eigenvalue. This is valid also for the problem (2.3),b).

Let us investigate the initial post-critical shell strain when $\lambda = 8.3$ and 7.89. The critical pressure determined experimentally in /5/ is 0.915 for the former shell and 1.010 for the latter /3/. A numerical analysis of the spectrum $\{p_n\}$ resulted in the following: $\lambda = 8.3$. $p_2 = 0{,}915$, $p_3 = 0.782$, $p_4 = 0.757$, $p_5 = 0.790$, $p_6 = 0.854$, $p_7 = 0.933$, $p_8 = 1{,}015$, $p_9 = 1{,}087$; $\lambda = 7.89$, $p_2 = 0.867$, $p_3 = 0.760$, $p_4 = 0.781$, $p_5 = 0.812$, $p_6 = 0.890$, $p_7 = 0.978$, $p_8 = 1.058$. It is seen that if $\lambda = 8.3$, then the spectrum consists of eight points, and if $\lambda = 7.89$, of seven points. The least eigenvalue 0.757 is achieved at $n = 4$ for the first shell, and at $n = 3$ for the second, where both these values are less than the corresponding experimental critical pressures. Let us take into account that the perturbing pressure is nonuniform in the experiment. We approximate the functional J_0 in (2.11) as follows:

$$J_0 = \int_0^1 g(r) r^3 [1 + a_1 \operatorname{th} M(m - r)] dr, \quad M = 10 \tag{3.1}$$

A numerical analysis showed that if $a_1 = 1$ and $m = 0.3$, then the bifurcation of the nonaxisymmetric modes for these shells is accomplished without snapping upon reaching the first eigenvalues since $\delta_2 = J / J_0$; the shells can hence sustain a pressure exceeding p_4 and p_3, respectively. In these cases, the nonaxisymmetric modes are realized in experiment if they are energetically suitable. For $\lambda = 8.3$, the bifurcation is accompanied by snapping only for $n = 9$, and for $\lambda = 7.89$ snapping occurs upon the solutions corresponding to the critical pressure of the goemetrically nonlinear theory in an axisymmetric formulation reaching the bifurcation point. If for $\lambda = 8.3$ and $m = 0.3$ there is a monotonic decrease in a_1 from $1.022 - \varepsilon$ (here $\varepsilon > 0$ is a small scalar parameter) to $0.633 - \varepsilon$, then snapping will occur sequentially for $n = 9, 8, 6, 5, 4, 2, 7, 3$. If $a_1 = 0.721 - \varepsilon$, then bifurcation is accompanied by snapping at the first eigenvalue $p_4 = 0.757$, and no nonaxisymmetric modes are observed in the static state.

An analogous analysis for $7.6 \leqslant \lambda \leqslant 20.7$ showed that the Koiter parameters corresponding to the rapidly oscillating waves, are as a rule more responsive to a change in a_1 than the parameters corresponding to slowly oscillating waves.

In a number of cases, axisymmetric equilibrium modes /6/ were observed in a precision experiment. A comparative analysis of the numerical results and the data in /6/ showed that the experimental values for the normal displacement agree qualitatively with the computation results. Thus, for $\theta = 0.2$, $h = 1.3$ mm, $a = 150$ mm, $\nu = 0.4$, and $p = 0.79$, the normal displacement w_1 at the pole of the shell is 0.225 mm, and measured in the experiment is 0.21 mm. For points at which the displacement takes on its maximum value, these quantities are 0.43 and 0.48 mm, respectively. For 90 mm $\leqslant r_1 \leqslant 150$ mm no discrepancy is detected between the computation and the precision experiment results. For points at which an abrupt change in the shell deflection is observed, the experimental values of the displacement exceed the computed values by $\sim 30\%$.

The strains ε_r and ε_φ were compared for the same shell for $p = 0.94$. The measurement results here lie $\sim 15\%$ either above or below the computed data, depending on the coordinates of the point under investigation.

Let us turn attention to an interesting fact detected in experiment /6/ and confirmed by numerical analysis. A sufficiently thin shell with a rigidly clamped edge is stretched in the circumferential direction in a small zone at the support contour.

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